DEFINIBILITY IN NORMAL THEORIES

BY

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ABSTRACT

This paper initiates an investigation which seeks to explain elementary definability as the classical results of mathematical logic (the completeness, compactness and Löwenheim-Skolem theorems) explain elementary logical consequence. The theorems of Beth and Svenonius are basic in this approach and introduce automorphism groups as a means of studying these problems. It is shown that for a complete theory T, the definability relation of Beth (or Svenonius) yields an upper semi-lattice whose elements (concepts) are interdefinable formulas of T (formulas having equal automorphism groups in all models of T). It is shown that there are countable models A of T such that two formulae are distinct (not interdefinable) in T if and only if they are distinct (have different automorphism groups) in A. The notion of a concept h being normal in a theory T is introduced. Here the upper semi-lattice of all concepts which define h is proved to be a finite lattice — anti-isomorphic to the lattice of subgroups of the corresponding automorphism group. Connections with the Galois theory of fields are discussed.

1. Introduction

In 1932 all the basic results concerning the notion of elementary logical deductions were available; namely, the theorems of Skolem-Löwenheim, Herbrand and Gödel. Four additional years of investigation of the notion of algorithm brought about the solution of Hilbert's decision problem of mathematics by Church and (independently and almost simultaneously) Turing. A few years later the compactness theorem for elementary logic must have been clear to several people.

At this time it might have seemed that most of the basic problems of elementary axiom systems were solved. A more careful observer however, upon reading the papers of Tarski [13, 14], might have wondered about the existence of general theorems which would explain elementary definability as the above theorems

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explain the basic properties of elementary logical consequence. One such theorem, the completeness, in the sense of definability, of elementary logic was proved by Beth in 1953 [1]. Also, around this time, Craig, in his doctoral dissertation, proved what is now known as the interpolation lemma (or separation lemma).

Craig's lemma [6] and Beth's theorem went unnoticed for a time. In 1956 the basic importance of Craig's lemma became apparent: first, it admits Beth's theorem as an easy corollary and second, it sounds more impressive in its model-theoretic version. About this time also, Robinson's consistency lemma [11], which is intimately related to Craig's lemma, appeared. The relationship between these definability results may be traced on various levels, i.e., there are stronger forms which may be proved by using model-operators like ultrapowers, etc. For a discussion of these various levels, see Buchi and Danhof [3].

In 1959 Svenonius [12] published a further result on elementary definability. Just as with the earlier results of Beth and Craig, logicians seem slow in recognizing Svenonius' theorem as a basic tool in the theory of definability, perhaps because it is not generally known to be available.

With the appearance of Klein's Erlangerprogramm in 1872 [10], it became apparent that automorphism groups are a most useful means of studying mathematical theories. In a more rigorous model-theoretic manner, these ideas have been discussed in the case of various elementary mathematical theories by Buchi and Wright (see [4, 5, 15]). It is not surprising that both the theorems of Beth and Svenonius are about the automorphism groups of the models of a theory. In Buchi [2], Beth's theorem is restated as a general result on relative categoricity and Svenonius' theorem is used to establish a basic result on the Galois group of normal concepts. Here these matters are carried out in detail and further results leading toward a theory of elementary definability are added.

2. Preliminaries

We assume familiarity with the notion of an *elementary theory* (*class*) $T = T(\boldsymbol{R})$ with primitives $\boldsymbol{R}$ and equality. We frequently identify a theory $T(\boldsymbol{R})$ with the class of models $\langle A, R\rangle$ of T (and assume this class to be closed under isomorphism). For a system A, $\tau(A)$ denotes the set of all sentences true in A (*elementary theory of* A). A theory $T(\boldsymbol{R})$ is *pseudo-elementary* if there is an elementary theory $T(\boldsymbol{R},\boldsymbol{S})$ such that $T(\boldsymbol{R})$ is the class of systems $\langle A, R\rangle$ for which there is an S with $\langle A, R, S\rangle \in T(\boldsymbol{R},\boldsymbol{S})$. For a system A, κA denotes the group of automorphisms of

A; $A \cong A'$ indicates that A and A' are isomorphic. A formula is called an *R'-condition* if the predicate R' (primitive or defined) is its only extralogical constant. For a system A, a one-one mapping ϕ between two subsets of A is called a *partial R'-automorphism* if any two sequences of elements which correspond to each other by ϕ satisfy the same R'-conditions. "$(\forall A)_T \cdots$" should be read as "for all models A of T, $\cdots$". An elementary formula $\Phi = \Phi(R, x_1, \cdots, x_n)$ of $T(R)$ defines an *elementary concept* of T as follows: For any system $A = \langle A, R \rangle$,

$$\Phi(\langle A, R\rangle) = \langle A, (\hat{x}_1, \cdots, \hat{x}_n)\Phi\rangle.$$

Thus, an elementary concept c of T ($c \in \overline{ec}(T)$) is a mapping defined by an elementary formula from a species into a (possibly different) species. Note that A and cA have the same domain. Moreover, for any such concept and any transformation ϕ of A we have:

i) $$\phi c A = c \phi A.$$

For the purpose of comparing elements of $\overline{ee}(T)$, we have the following two quasi-orders:

1) $c \leqq_1 d(T) \,.\!\equiv.\, (\forall A, A')_T\, dA = dA' \rightarrow cA = cA'$

2) $c \leqq_2 d(T) \,.\!\equiv.\, (\forall A)_T\, \kappa dA \subseteq \kappa cA$.

It is easy to see that $c \leqq_1 d(T)$ implies $c \leqq_2 d(T)$. For T pseudo-elementary, Craig's extension [7] of Beth's theorem may be stated as follows:

ii) $c \leqq_1 d(T) \,.\!\equiv.$ there is an elementary concept d' such that

$$(\forall A)_T\, cA = d' dA.$$

Similarly, with T again pseudo-elementary, Svenonius' theorem [12] may be generalized to:

iii) $c \leqq_2 d(T) \,.\!\equiv.$ there are elementary concepts $d_1, \cdots, d_n$ such that

$$(\forall A)_T\, (cA = d_1 dA \vee \cdots \vee cA = d_n dA).$$

Note in particular that if T is a complete elementary theory, then for all c, $d \in \overline{ec}(T)$, $c \leqq_1 d(T) \,.\!\equiv.\, c \leqq_2 d(T)$. In the sequel, we restrict our attention to complete theories and let $\leqq$ denote $\leqq_1$ (or equivalently $\leqq_2$). For $h \in \overline{ec}(T)$, let $\overline{ec}(T, h) = \{c \in \overline{ec}(T);\, h \leqq c(T)\}$. $\leqq$ is a quasi-order on $\overline{ec}(T, h)$ and if $\sim$ is defined by $c \sim d \,.\!\equiv.\, c \leqq d(T) \wedge d \leqq c(T)$, then $\sim$ is an equivalence relation on $\overline{ec}(T, h)$. $ec(T, h)$ will denote the set of equivalence classes (relative to the relation $\sim$). $\leqq$ induces a partial order on these classes. The partially ordered set has $(0-)$ the concept h and $(1-)$ the primitive concept $\boldsymbol{R}$. Hereafter, we often identify a concept with its equivalence class. For $c,\ c' \in ec(T,h)$ defined by $\Phi(x_1,\cdots,x_n)$ and $\Phi'(y_1,\cdots,y_m)$, we let $c \otimes c' = (\hat{x}_1 \cdots \hat{x}_n \hat{y}_1 \cdots \hat{y}_m)\ (\Phi \wedge \Phi')$. $\otimes$ is a l.u.b. in $ec(T,h)$ and consequently $ec(T,h)$ is an upper semi-lattice with 0 and 1. Note that if the concept '$=$' is defined by the formula $x_1 = x_2$, then for any concept c, '$=$' $\leqq c(T)$. We denote $ec(T,$ '$=$'$)$ by $ec(T)$.

For a structure A of T, we write $c \leqq d(A)$ if $\kappa dA \subseteq \kappa cA$. Note $c \leqq d(T)$ implies $c \leqq d(A)$. By imitating the above construction, we get, for each model A of T and each $h \in \overline{ec}(T)$, an upper semi-lattice $ec(A,h)$—a subsystem of $ec(T,h)$.

It can be shown that $ec(T)$ need not be a lattice. For example, let

$$T = \tau\langle A, f_1, f_2, a\rangle$$

where $a \in A$, f_i is a partial one-one unary function from $A_i = \{f_i^n(a);\ n \geqq 0\}$ onto $A_i - \{a\}$ for $i = 1,2$, and $f_1^n(a) = f_2^m(a)$ iff $n = m$ and n is even (i.e., $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \{f_i^n(a);\ n \text{ even}\}$). One can then show g.l.b. $\{f_1(x) = y, f_2(x) = y\}$ does not exist (such a g.l.b. would be definable from each of f_1 and f_2).

Several questions suggest themselves at this point: What is the relation between $ec(T)$ and the theory T? When is $ec(T,h)$ finite?, etc. In the following section we describe a sufficient condition for the finiteness of $ec(T,h)$.

3. The following lemma expands only slightly the basic result used in extending Beth's theorem to Svenonius' theorem and the proof employs the ideas of that proof (see Svenonius [12]).

LEMMA 1. *Given elementarily equivalent systems $\boldsymbol{A}$ and $\boldsymbol{A}'$, concepts c and c' and partial automorphisms ϕ (ϕ') of $c\boldsymbol{A}$ ($c'\boldsymbol{A}'$), there is an elementary extension $\boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{A}'$ and partial automorphisms σ (σ') of $c\boldsymbol{B}$ ($c'\boldsymbol{B}$) which extend ϕ (ϕ') such that the domain (or range) of σ (σ') is the image of $A(A')$ in B.*

Our first theorem shows that for T complete, the study of $ec(T,h)$ reduces to that of $ec(A,h)$ for certain countable models of A of T.

THEOREM 1. *If T is complete and $h \in ec(T)$, there is a countable model $\boldsymbol{A}$ of T such that $ec(T,h) = ec(\boldsymbol{A},h)$.*

OUTLINE OF PROOF. $ec(T,h)$ is countable (its elements are represented by formulas of T). Let $\{\alpha_i\}_{i<\omega}$ ennumerate all pairs $\langle c,c'\rangle \in (ec(T,h))^2$ such that $c < c'(T)$. Then for each $i < \omega$ and $\alpha_i = \langle c_i, c_i'\rangle$ there is a model $\boldsymbol{A}_i$ of T such that $\kappa c_i'\ \boldsymbol{A}_i \subset \kappa c_i \boldsymbol{A}_i$; moreover we may assume $\boldsymbol{A}_i$ countable. Now using Lemma 1 one constructs models $\boldsymbol{B}_n$ of T and partial mappings ϕ_i^n such that $\boldsymbol{B}_0 = \boldsymbol{A}_0$, $\boldsymbol{B}_{n+1}$

is an elementary extension of B_n and A_{n+1}, and $\phi_i^{n+1}(i = 0, \cdots n+1)$ is a partial automorphism of $c_i B_{n+1}$ but not of $c_i' B_{n+1}$. As n is even (odd) the domain (range) of ϕ_i^{n+1} $(i \leqq n)$ is the image of B_n in B_{n+1} and the domain (range) of ϕ_{n+1}^{n+1} is the image of A_{n+1} in B_{n+1}. $A = \cup B_n$ is now the desired system since for $\phi_i = \cup \phi_i^n$ we have $\phi_i \in \kappa c_i A / \kappa c_i' A$ for $i < \omega$. Thus $c < d(T) \ .\equiv.\ c < d(A)$ and consequently $ec(T,h) = ec(A,h)$.

DEFINITION. The theory T is said to be *normal relative the elementary concept* h $(h \lhd T)$ if:

$$(\forall A, A')_T A \cong A' \wedge hA = hA' \ .\!\rightarrow\!.\ \kappa A = \kappa A'.$$

Models A and A' such that $A \cong A'$ and $hA = hA'$ (i.e., there is an automorpism of $hA = hA'$ which takes A onto A') are called *h-conjugate* and we write $A \simeq A'(h)$.

LEMMA 2. $h \lhd T .\equiv. (\forall A)_T \kappa A \lhd \kappa hA$ (i.e., *κA is a normal subgroup of κhA*).

PROOF. Note that by definition,

$$h \lhd T .\equiv. (\forall A, A')_T A \simeq A'(h) \ .\!\rightarrow\!.\ \kappa A = \kappa A',$$

or equivalently, $(\forall A)_T\ \phi hA = hA \wedge \psi A = A \ .\!\rightarrow\!.\ \psi\phi A = \phi A$. However, this last condition may be restated as $(\forall A)_T\ \phi \in \kappa hA \wedge \psi \in \kappa A \ .\!\rightarrow\!.\ \phi^{-1}\psi\phi \in \kappa A$.

THEOREM 2. *If $h \lhd T$, there is a number n and elementary concepts $d_1, \cdots, d_n$ such that for any model A of T,*

$$A \simeq A'(h) \ .\!\rightarrow\!.\ A' = d_1 A \vee \cdots \vee A' = d_n A$$

and in particular, $G_h(A) = \kappa hA / \kappa A$ has order $\leqq n$.

PROOF. Since $h \lhd T$, we have

$$\text{a)} \qquad (\forall A)_T A \simeq A'(h) \ .\!\rightarrow\!.\ \phi A = A \rightarrow \phi A' = A'.$$

Recall that $A \simeq A'(h)$ means $hA = hA' \wedge (\exists\phi)\ \phi A = A'$. Since h is elementary, this is a pseudo-elementary proposition. Since $T = T(\boldsymbol{R})$ is elementary, it follows that $A \in T \wedge A \simeq A'(h)$ is a pseudo-elementary theory $T'(\boldsymbol{R}, \boldsymbol{R}')$. Now (a) is just the other assumption of (iii) (Svenonius' theorem) and hence there are elementary concepts $d_1, \cdots, d_n$ such that

$$A \in T \wedge A \simeq A'(h) \, .\!\rightarrow\!. \, A' = d_1 A \vee \cdots \vee A' = d_n A.$$

This establishes the first part of Theorem 2.

To see that $G_h(A)$ is finite, let $A \in T$. Since, by the above, there are at most n different h-conjugates $A_1, \cdots, A_n$ of A, we have

b) $$\phi \in \kappa h A \rightarrow \phi A = A_1 \vee \cdots \vee \phi A = A_n.$$

Let σ, $\rho \in \kappa h A$. Then by (b), we have

c) $$\sigma^{-1}\rho \in \kappa A \, .\!\equiv\!. \, \sigma A = \rho A = A_1 \vee \cdots \vee \sigma A = \rho A = A_n.$$

From (c), one sees the relation $\sigma^{-1}\rho \in \kappa A$ is an equivalence relation on $\kappa h A$, of index $\leqq n$. Moreover, it is the congruence on $\kappa h A$ relative to the normal subgroup κA. Therefore, $G_n(A)$ has $\leqq n$ members.

We give two examples where $h \lhd T$. T_1 has n unary predicate letters $R_1, \cdots, R_n$. Axioms for T_1 insure that in any model of T_1, the interpretations of $R_1, \cdots, R_n$ partition the domain of the model into n disjoint equivalence classes. Let h be given by the formula which defines the resulting equivalence relation. It is easy to show $h \lhd T_1$. Each model of T_1 has $\leqq n!$ h-conjugates and equality holds if all of the equivalence classes have the same cardinality. In the latter case $G_h(A)$ is just the symmetric group on n objects.

As a second example, let F be a finite (algebraic) extension of its prime subfield and $G = F(\theta_1, \cdots, \theta_n)$ a finite extension of F. Let T_2 be all sentences true in $\mathbf{G} = \langle G, F(G), \theta_1, \cdots, \theta_n \rangle$ where $F(G)$ consists of $0, 1, +, \cdot$ and the elements needed to generate F. For $\mathbf{A} = \langle A, F(A), a_1, \cdots, a_n \rangle$ a model of T_2, let $h\mathbf{A} = \langle A, F(A) \rangle$. Using basic facts about fields, we again have $h \lhd T_2$.

THEOREM 3. *If $h \lhd \tau(A)$, then there is an order anti-isomorphism from $ec(A,h)$ onto the lattice of subgroups of $G_h(A)$. Consequently, applying Theorem 2, $ec(A,h)$ is a finite lattice.*

PROOF. For $c \in ec(A,h)$, $c \rightarrow \kappa c A / \kappa A \subseteq G_h(A)$. This correspondence is clearly one-one, for $\kappa c A / \kappa A = \kappa c' A / \kappa A \, .\!\equiv\!. \, \kappa c A = \kappa c' A \, .\!\equiv\!. \, c = c'$. Also, $c \leqq c'(A) \, .\!\equiv\!. \, \kappa c' A \subseteq \kappa c A \, .\!\equiv\!. \, \kappa c' A / \kappa A \subseteq \kappa c A / \kappa A$. We are to show that the mapping is onto. By Theorem 2, there are elementary concepts $d_1, \cdots, d_n$ such that

1) $$A \cong A' \wedge hA = hA' \, .\!\rightarrow\!. \, A' = d_1 A \vee \cdots \vee A' = d_n A.$$

Moreover, we may take $d_iA = \phi_iA$ where $G_h(A) = \{\phi_i; i = 1,\cdots,n\}$. One also has (see (i)) that if $\phi_i\phi_j = \phi_k$, then $d_jd_iA = d_kA$ and if $\phi^{-1} = \phi_j$, then $d_jd_iA = d_id_jA = A$ (we take $d_1A = A$, i.e., ϕ_1 is the identity). Thus, $D = \{d_i; i = 1,\cdots,n\}$ restricted to $\{d_iA; i = 1,\cdots,n\}$ forms a group anti-isomorphic to $G_h(A)$.

For d_i, $d_j \in D$, $\sim d_i$, $d_i \wedge d_j$ and $d_i \vee d_j$ are defined in the obvious manner. For each $f: \{d_2,\cdots,d_n\} \to \{d_2, \sim d_2,\cdots,d_n, \sim d_n\}$ such that $f(d_j) \in \{d_j, \sim d_j\}$ for $j = 2,\cdots,n$, let

2) $$d_i^f = d_i \wedge (\wedge_{j=2,\dots,n} f(d_j)d_i); \qquad i \leqq n.$$

We have,

3) $$a \in d_jd_iA \;.\equiv.\; \phi_i^{-1}\ (a) \in d_jA.$$

From (3), $d_i^f(A) = 0 \;.\equiv.\; d_j^f(A) = 0$. We say $f \neq 0$ if $d_1^f(A) \neq 0$. For $i = 1,\cdots,n$; $d_i^* = \otimes_{f \neq 0}\ d_i^f$ and for $H \subseteq G_h(A)$,

4) $$d_H = \left(\bigvee_{\phi_i \in H} d_i^*\right) \otimes h.$$

Note $\kappa d_HA \subseteq \kappa hA \to d_H \in ec(A,h)$. We assert $H = \kappa d_HA/\kappa A$. If $\sigma \in H$, then $\sigma = \phi_k$ for some $k \leqq n$. Also, $\sigma d_HA = d_HA$ since if $\phi_i \in H$, $\sigma d_i^*A = \phi_k d_i^*A = d_i^* d_kA = (d_id_k)^*A = d_j^*A$ for $\phi_j = \phi_k\phi_i \in H$. Thus, $H \subseteq \kappa_{d_H}A\, d\kappa A$. For the converse, note that $d_j^*A \cap d_i^*A = 0$ for $i \neq j$. If $\sigma \in \kappa d_HA/\kappa A$, then $\sigma = \phi_j$ for some $j \leqq n$. Now for $a \in d_1^*A$,

$$\sigma a \in \left(\bigvee_{\phi_i \in H} d_i^*\right)$$

i.e., $\sigma a \in d_i^*A$ for some $\phi_i \in H$. By (3), $a \in d_k^*A$ where $\phi_k = \phi_j^{-1}\phi_i$ and hence $k = 1$ by the disjointness property noted above. Thus, $\phi_j = \phi_i$ and $\sigma = \phi_j \in H$. This proves the theorem.

Combining Theorems 1 and 3, we have;

THEOREM 4. *If $h \lhd T$ (T complete), then $ec(T,h)$ is anti-isomorphic to the lattice of subgroups of $G_h(A)$ for some countable model A of T (and hence is a finite lattice).*

Returning to the earlier example T_2 from fields, since $h \lhd T_2$, Theorem 3 applies here. Moreover, if G is assumed to be the splitting field of a separable polynomial over F, then $G = F(\theta)$, a simple extension of F, and $G_h(G)$ is just the group of automorphisms of G fixing F. In this case also, there is a one-one correspondence between $ec(\boldsymbol{G}, h)$ and subfields J, $F \subseteq J \subseteq G$. For if $F \subseteq J \subseteq G$, then $J = F(\delta)$ for some $\delta \in G$. We let $c_J\boldsymbol{G} = \langle G, F(G), \delta \rangle$; then $\kappa c_J \boldsymbol{G}$ is the group of automorphisms of G over J. Also, if $c_J = c_{J'}$, then $J = J'$. Conversely, given $c \in ec(\boldsymbol{G}, h)$, let $J = \{a \in G;\ \phi a = a$ for all $\phi \in \kappa c G = H\}$. Then $c_J \leqq c(\boldsymbol{G})$ and from the proof of Theorem 3, we may write $c = \bigvee_H (x = \phi_i(\theta)) \otimes h$. Now in T_2, $\bigvee_H (x = \phi_i(\theta))$ $. \equiv . f(x) = 0$; it follows that the coefficients of $f(x)$ are in J. Any element of $\kappa c_J \boldsymbol{G}$ fixes $f(x)$ and hence fixes c. Thus, $c \leqq c_J(\boldsymbol{G})$. It follows that there is a one-one correspondence between the subgroups of $\kappa h(G)$ and subfields $J, F \subseteq J \subseteq G$. (cf. Fundamental theorem of Galois theory).

Remark. In an attempt to keep the presentation as smooth as possible, our definition of elementary concept in §2 was less general than it might have been. As an example of how this notion might be extended, for $\Phi = \Phi(\boldsymbol{R}, \boldsymbol{S}, x_1, \cdots, x_n)$ a formula of $T(\boldsymbol{RS})$, we can let $\Phi(\langle A, R, S \rangle) = \langle A, R, (\hat{x}_1, \cdots, \hat{x}_n)\Phi \rangle$. This version subsumes the earlier notion and the theorems stated above remain intact. In this setting the restriction in example 2 that F be a finite extension of its prime field can be removed; F can now be any field.

We conclude with a final observation reminiscent of the Galois groups of fields. Let $h \lhd T$, A be a model of T and $c \in ec(T, h)$. Then for $\sigma \in \kappa h A$, $\kappa c \sigma A = \sigma(\kappa c A)\sigma^{-1}$. Now $\kappa c A = \sigma(\kappa c A)\sigma^{-1}$ for all $\sigma \in \kappa h A$ $. \equiv .$ $\kappa c A \lhd \kappa h A$ $. \equiv .$ $A \simeq A'(h)$ $. \rightarrow .$ $\kappa c A = \kappa c A'$. Accordingly, we write

$$h \lhd c(T) \;.\!\equiv\!.\; (\forall A, A')_T A \simeq A'(h) \;.\!\rightarrow\!.\; \kappa c A = \kappa c A'.$$

From the above, $h \lhd c(T) \;.\!\equiv\!.\; (\forall A)_T\ \kappa c A \lhd \kappa h A \;.\!\equiv\!.\; (\forall A)_T (\forall \sigma)_{\kappa h \bar{A}}\ \kappa c \sigma A = \kappa c A$. Now if $\kappa c A \lhd \kappa h A$, then $\kappa c A / \kappa A \lhd \kappa h A / \kappa A$. In this case, by the law of homomorphism for groups,

$$\kappa h A / \kappa c A \cong (\kappa h A / \kappa A) / (\kappa c A / \kappa A).$$

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